DPP No.- 49 to 52

Time: 30 Min. **M.M.: 24** **DPP. NO.-49**

[SINGLE CORRECT CHOICE TYPE] [8 × 3 = 24]

Q.1 Let $f(x) = \sin^3 x + \sin^3\left(x + \frac{2\pi}{3}\right) + \sin^3\left(x + \frac{4\pi}{3}\right)$ then the primitive of $f(x)$ w.r.t. x is

(A) $-\frac{3\sin 3x}{4} + C$ (B) $-\frac{3\cos 3x}{4} + C$ (C) $\frac{\sin 3x}{4} + C$ (D) $\frac{\cos 3x}{4} + C$

where C is an arbitrary constant.

Q.2 If equation of tangent drawn to the curve $y = \left|\log_2 |x|\right|$ at $x = \frac{-1}{3}$ is $px + y\,ln(q) + ln(3e) = 0$ then $(p + q)$ is equal to

(A) 3 (B) $\frac{5}{2}$ (C) 5 (D) $\frac{7}{2}$

Q.3 If $f(x) = \lim_{t\to 0} \tan^{-1}\left(\frac{e^{xt} - 1}{t}\right)$, then $\lim_{x\to 0} \frac{f(x) - x}{x^3}$ is equal to

(A) $\frac{1}{2}$ (B) $\frac{-1}{2}$ (C) $\frac{1}{3}$ (D) $\frac{-1}{3}$

Q.4 Let f(x) be a differentiable function such that $f(x) + 2f(-x) = \sin x$ for all $x \in R$.

The value of $f'\left(\frac{\pi}{4}\right)$ is equal to

(A) $\frac{1}{\sqrt{2}}$ (B) $\frac{-1}{\sqrt{2}}$ (C) $\sqrt{2}$ (D) $-\sqrt{2}$

Q.5 If the equation $\sin^{-1}(4x - x^2 - 5) + \cos^{-1}(4x - x^2 - 5) + \lambda x = 0$ has a real solution, then λ is equal to

(A) $\frac{-\pi}{4}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{2}$ (D) $\frac{-\pi}{2}$

Q.6 The graph of function $f(x) = \frac{x^5}{20} - \frac{x^4}{12} + 5$ has

(A) no local extremum, one point of inflection.
(B) two local maximum, one local minimum, two point of inflection.
(C) one local maximum, one local minimum, one point of inflection.
(D) one local maximum, one local minimum, two point of inflection.

Q.7 If the curves $x^{2/3} + y^{2/3} = c^{2/3}$ and $\mathbf{(x^2/a^2) + (y^2/b^2) = 1}$ touches each other then

(A) $a + b = c$ (B) $a - b = c$ (C) $a + 2b = c$ (D) $2a - b = c$

Q.8 The value of $\sum_{r=1}^{\infty} \cot^{-1}\left(\frac{r^2}{2} + \frac{15}{8}\right)$ is equal to

(A) $\tan^{-1}1$ (B) $\tan^{-1}2$ (C) $\tan^{-1}3$ (D) $\tan^{-1}4$

Time: 30 Min. **M.M.: 24** **DPP. NO.-50**

[SINGLE CORRECT CHOICE TYPE] [5 × 3 = 15]

Q.1 The slope of tangent drawn to the curve $f(x) = \sin x - \int_0^x (x-t)f(t)\,dt$ at $x = 0$, is equal to

(A) 1 (B) 2 (C) 3 (D) 4

Q.2 Let $f : R \to R$ be any function. Define $g : R \to R$ by $g(x) = |f(x)|$ for all x. Then g is
(A) onto if f is onto (B) one one if f is one one
(C) continuous if f is continuous (D) differentiable if f is differentiable.
[JEE 2000, Screening]

Q.3 Let $f(x) = mx - 1 + \frac{1}{x}$. Then the smallest value of the constant m such that $f(x) \geq 0$ for every $x > 0$ is

(A) $\frac{1}{8}$ (B) $\frac{1}{4}$ (C) $\frac{1}{2}$ (D) 1

Q.4 If the two distinct linear functions which map the interval $[-1, 1]$ onto $[0, 2]$ are $f(x) = ax + b$ and $g(x) = cx + d$ then $(a + b + c + d)$ is equal to
(A) 1 (B) 2 (C) 3 (D) 0

Q.5 Let $f : (-3, 3) \to R$ be a differentiable function with $f(0) = -2$ and $f'(0) = -1$.
and $g(x) = (f(3f(x)+6))^3$. Then $g'(0)$ is equal to
(A) 0 (B) 9 (C) 36 (D) – 36

[COMPREHENSION TYPE] [3 × 3 = 9]

Paragraph for Question 6 to 8

Let P(x) be a polynomial of degree 4, vanishes at $x = 0$. Given $P(-1) = 55$ and P(x) has relative maximum / relative minimum at $x = 1, 2, 3$.

Q.6 Area of triangle formed by extremum points of P(x), is

(A) $\frac{1}{2}$ (B) $\frac{1}{4}$ (C) $\frac{1}{8}$ (D) 1

Q.7 The value of definite integral $\int_{-1}^{1} (P(-x) + P(x))\,dx$, is

(A) $\frac{252}{15}$ (B) $\frac{452}{15}$ (C) $\frac{652}{15}$ (D) $\frac{752}{15}$

Q.8 Which one of the following statement is correct?
(A) P (x) has two relative maximum points and one relative minimum point.
(B) Range of P(x) contains 9 negative integers.
(C) Sum of real roots of $P(x) = 0$ is 5.
(D) P (x) has exactly one inflection point.

Time: 30 Min. **M.M.: 20** **DPP. NO.-51**

[MULTIPLE CORRECT CHOICE TYPE] [5 × 4 = 20]

Q.1 Let $I = \int_0^1 \sqrt{\frac{1+\sqrt{x}}{1-\sqrt{x}}}\,dx$ and $J = \int_0^1 \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\,dx$, then the correct statement is

(A) $I + J = 2$ (B) $I - J = \pi$ (C) $I = \frac{2+\pi}{2}$ (D) $J = \frac{4-\pi}{2}$

Q.2 If tangents are drawn from a point P (2, 0) to the curve $\sqrt{1+y^2} = \frac{x}{3}$ which meet the curve at A and B, then

(A) Acute angle between the tangents is $\tan^{-1}\left(\frac{\sqrt{5}}{2}\right)$

(B) $AB = \sqrt{5}$

(C) Area of $\Delta PAB = \frac{5\sqrt{5}}{4}$

(D) ΔPAB is equilateral triangle.

Q.3 The function, $f(x) = \max\{(1-x), (1+x), 2\}, x \in (-\infty, \infty)$ is :
(A) Continuous at all points
(B) Differentiable at all points
(C) Differentiable at all points except at $x = 1$ & $x = -1$
(D) Continuous at all points except at $x = 1$ & $x = -1$, where it is discontinuous. [JEE'95]

Q.4 Let f be a continuous function on R and satisfies $f(x) = e^x + \int_0^1 e^x f(t)\,dt$,

then which of the following is(are) correct?
(A) $f(0) < 0$ (B) $f(x)$ is decreasing function on R
(C) $f(x)$ is an increasing function on R (D) $\int_0^1 f(x)\,dx > 0$

Q.5 Let $h(x) = \min\{x, x^2\}$, for every real number of x. Then :
(A) h is cont. for all x (B) h is diff. for all x
(C) $h'(x) = 1$, for all $x > 1$ (D) h is not diff. at two values of x. [JEE'98]

Time: 30 Min. **M.M.: 25** **DPP. NO.-52**

[INTEGER TYPE] [5 × 5 = 25]

Q.1 Let d be the number of integers in the range of the function $f(x) = \begin{cases} 4, & \text{if } -4 \le x < -2 \\ |x|, & \text{if } -2 \le x < 7 \\ \sqrt{x}, & \text{if } 7 \le x < 14 \end{cases}$.

Also roots of $P(x) = x^2 + mx - 4m + 20$ are α and β. If $\alpha < \frac{d-3}{4} < \frac{d-3}{2} < \beta$ and the smallest integral value of m is k, then find the value of $(k-5)$.

Q.2 If y is a function of x and $\ln(x+y) = 2xy$, then find the value of $y''(0)$.

Q.3 Let $f(x) = \frac{1}{e^x + 8e^{-x} + 4e^{-3x}}$ and $g(x) = \frac{1}{e^{3x} + 8e^{x} + 4e^{-x}}$. If $\int \left(f(x) - 2g(x)\right)dx = h(x) + c$, where c is constant of integration and $\lim\limits_{x\to\infty} h(x) = \frac{\pi}{4}$, then find the value of $2\tan\left(2h(0)\right)$.

Q.4 If $\lim\limits_{n\to\infty} \frac{e\left(1-\frac{1}{n}\right)^n - 1}{n^{\alpha}}$ exists and is equal to non-zero constant c, then find the value of $12(c-\alpha)$.

Q.5 Let $f(x) = 12\left(\frac{e^{3x} - 3e^{x}}{e^{2x} - 1}\right)$ be defined for $x > 0$ and $g(x)$ be the inverse of $f(x)$.

If $\int_8^{27} g(x)\,dx = a\ln 3 - b\ln 2 - c$, then find the value of $a - (b+c)$.

DPP No.- 49 to 52

Solutions

Time: 30 Min. **M.M.: 24** **DPP. NO.-49**

[SINGLE CORRECT CHOICE TYPE] [8 × 3 = 24]

Q.1 Let $f(x) = \sin^3 x + \sin^3\left(x + \frac{2\pi}{3}\right) + \sin^3\left(x + \frac{4\pi}{3}\right)$ then the primitive of $f(x)$ w.r.t. x is

(A) $-\frac{3\sin 3x}{4} + C$ (B) $-\frac{3\cos 3x}{4} + C$ (C) $\frac{\sin 3x}{4} + C$ (D*) $\frac{\cos 3x}{4} + C$

where C is an arbitrary constant.

[Sol.$_{57/inde/SC}$ Note that $\sin x + \sin\left(x + \frac{2\pi}{3}\right) + \sin\left(x + \frac{4\pi}{3}\right) = 0$

$\Rightarrow \quad \sin^3 x + \sin^3\left(x + \frac{2\pi}{3}\right) + \sin^3\left(x + \frac{4\pi}{3}\right) = -\frac{3}{4}\sin 3x \qquad (a+b+c = 0 \Rightarrow a^3+b^3+c^3 = 3abc)$

$\therefore \quad -\frac{3}{4}\int \sin 3x\, dx = \frac{\cos 3x}{4} + C$ **Ans.**]

Q.2 If equation of tangent drawn to the curve $y = \left|\log_2 |x|\right|$ at $x = \frac{-1}{3}$ is $px + y\,ln(q) + ln(3e) = 0$ then $(p + q)$ is equal to

(A) 3 (B) $\frac{5}{2}$ (C) 5 (D*) $\frac{7}{2}$

[Sol.$_{344(tn)/aod/SC}$ at $x = -1/3$, $y = \log_2 3$

$y = \left|\log_2 |x|\right| = -\log_2(-x)$, $\qquad -1 < x < 0$

$\frac{dy}{dx} = \frac{-1}{ln 2} \cdot \frac{(-1)}{(-x)} = \frac{-1}{x\, ln 2}$

$\left.\frac{dy}{dx}\right|_{x=-1/3} = \frac{3}{ln 2}$

$\therefore$ equation of tangent at $(-1/3, \log_2 3)$

$y - \log_2 3 = \frac{3}{ln 2}\left(x + \frac{1}{3}\right)$

$y\, ln2 - ln3 = 3x + 1$

$3x - y\, ln 2 + 1 + ln3 = 0$

$px + y\, ln(q) + ln(3e) = 0$

$\therefore p = 3,\ q = \frac{1}{2} \Rightarrow p + q = \frac{7}{2}$]

Q.3 If $f(x) = \lim_{t\to 0} \tan^{-1}\left(\frac{e^{xt}-1}{t}\right)$, then $\lim_{x\to 0} \frac{f(x)-x}{x^3}$ is equal to

(A) $\frac{1}{2}$ (B) $\frac{-1}{2}$ (C) $\frac{1}{3}$ (D*) $\frac{-1}{3}$

[Sol.$_{327/lcd/SC}$ $f(x) = \tan^{-1} x$

$\therefore \lim_{x\to 0}\left(\frac{\tan^{-1}x - x}{x^3}\right) = \frac{-1}{3}$ **Ans.**]

Q.4 Let $f(x)$ be a differentiable function such that $f(x) + 2f(-x) = \sin x$ for all $x \in R$.

The value of $f'\left(\frac{\pi}{4}\right)$ is equal to

(A) $\frac{1}{\sqrt{2}}$ (B*) $\frac{-1}{\sqrt{2}}$ (C) $\sqrt{2}$ (D) $-\sqrt{2}$

[Sol.$_{208/mod/SC}$ $f'(x) - 2f'(-x) = \cos x$ ....(1)

Put $x = \frac{\pi}{4}$; $f'\left(\frac{\pi}{4}\right) - 2f'\left(\frac{-\pi}{4}\right) = \frac{1}{\sqrt{2}}$ ....(2)

Put $x = \frac{-\pi}{4}$; $2f'\left(\frac{-\pi}{4}\right) - 4f'\left(\frac{\pi}{4}\right) = \frac{2}{\sqrt{2}}$ ....(3)

$\therefore$ Adding (2) and (3), we get

$-3f'\left(\frac{\pi}{4}\right) = \frac{3}{\sqrt{2}} \Rightarrow f'\left(\frac{\pi}{4}\right) = \frac{-1}{\sqrt{2}}$. **Ans.**]

Q.5 If the equation $\sin^{-1}(4x - x^2 - 5) + \cos^{-1}(4x - x^2 - 5) + \lambda x = 0$ has a real solution, then λ is equal to

(A*) $\frac{-\pi}{4}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{2}$ (D) $\frac{-\pi}{2}$

[Sol.$_{177/itf/SC}$ We have, $\sin^{-1}\left(-1-(x-2)^2\right) + \cos^{-1}\left(-1-(x-2)^2\right) + \lambda x = 0$ ... (i)

$\therefore$ We must have $x = 2$. So, we get $\frac{-\pi}{2} + \pi + 2\lambda = 0 \Rightarrow \lambda = \frac{-\pi}{4}$. **Ans.**]

Q.6 The graph of function $f(x) = \frac{x^5}{20} - \frac{x^4}{12} + 5$ has

(A) no local extremum, one point of inflection.
(B) two local maximum, one local minimum, two point of inflection.
(C*) one local maximum, one local minimum, one point of inflection.
(D) one local maximum, one local minimum, two point of inflection.

[Sol.$_{362/aod/SC}$ $f(x) = \frac{x^2}{20} - \frac{x^4}{12} + 5$

$$f'(x) = \frac{x^4}{4} - \frac{x^3}{3} = \frac{x^3}{12}(3x-4)$$

$$\Rightarrow f''(x) = x^3 - x^2 = x^2(x-1)$$

$-\infty$ + 0 – 4/3 + $+\infty$
sign of f '(x)

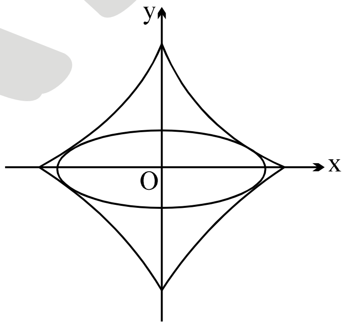
$-\infty$ – 0 – 1 + $+\infty$
sign of f "(x)

Now, verify alternative]

Q.7 If the curves $x^{2/3} + y^{2/3} = c^{2/3}$ and $\mathbf{(x^2/a^2) + (y^2/b^2) = 1}$ touches each other then
(A*) a + b = c (B) a – b = c (C) a + 2b = c (D) 2a – b = c

[Sol.$_{358/aod/SC}$ $x^{1/3} + y^{2/3} = c^{4/3}$ $\qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0 \qquad \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$



$$\frac{dy}{dx} = -\frac{y_1^{1/3}}{x_1^{1/3}}$$

$$y - y_1 = -\frac{y_1^{1/3}}{x_1^{1/3}}(x - x_1)$$

$$yx_1^{1/3} - y_1x_1^{1/3} = -y_1^{1/3}x + x_1y_1^{1/3}$$

$$y_1^{1/3}x + x_1^{1/3}y = x_1^{1/3}y^{1/3}\left(x_1^{1/3} + y_1^{2/3}\right)$$

$$\left.\begin{array}{l}\frac{x}{x_1^{1/3}} + \frac{y}{y_1^{1/3}} = c^{2/3} \\ \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1\end{array}\right]\text{identical}$$

$$\frac{a}{c^{1/3}} + \frac{b}{c^{1/3}} = c^{2/3}$$

$$\frac{x_1^{4/3}}{a^2} = \frac{y_1^{4/3}}{b^2} = \frac{1}{c^{2/3}}$$

$$x_1^{4/3} = \frac{a^2}{c^{2/3}}$$

$$\left.\begin{array}{l}x_1^{2/3} = \frac{a}{c^{1/3}} \\ \text{||| ly} \quad y_1^{2/3} = \frac{b}{c^{1/3}}\end{array}\right] \Rightarrow x_1^{2/3} + y_1^{2/3} = \frac{a+b}{c^{1/3}} \Rightarrow c^{2/3} = \frac{a+b}{c^{1/3}} \Rightarrow a + b = c \text{]}$$

Q.8 The value of $\sum_{r=1}^{\infty} \cot^{-1}\left(\frac{r^2}{2}+\frac{15}{8}\right)$ is equal to

(A) $\tan^{-1}1$ (B) $\tan^{-1}2$ (C) $\tan^{-1}3$ (D*) $\tan^{-1}4$

[Sol.$_{125/itf/SC}$ Sum $= \sum_{r=1}^{\infty} \cot^{-1}\left(\frac{r^2}{2}+\frac{15}{8}\right) = \sum_{r=1}^{\infty} \tan^{-1}\left(\frac{1}{\frac{r^2}{2}+\frac{15}{8}}\right) = \sum_{r=1}^{\infty} \tan^{-1}\left(\frac{2}{r^2+\frac{15}{4}}\right)$

$$= \sum_{r=1}^{\infty} \tan^{-1}\left(\frac{2}{4+r^2-\frac{1}{4}}\right) = \sum_{r=1}^{\infty} \tan^{-1}\left(\frac{2}{4+\left(r+\frac{1}{2}\right)\left(r-\frac{1}{2}\right)}\right) = \sum_{r=1}^{\infty} \tan^{-1}\left(\frac{\frac{1}{2}}{1+\left(\frac{r+\frac{1}{2}}{2}\right)\left(\frac{r-\frac{1}{2}}{2}\right)}\right)$$

$$= \sum_{r=1}^{\infty} \tan^{-1}\left\{\frac{\frac{\left(r+\frac{1}{2}\right)}{2}-\frac{\left(r-\frac{1}{2}\right)}{2}}{1+\frac{\left(r+\frac{1}{2}\right)}{2}\cdot\frac{\left(r-\frac{1}{2}\right)}{2}}\right\} = \lim_{n\to\infty} \sum_{r=1}^{n}\left(\tan^{-1}\left(\frac{r+\frac{1}{2}}{2}\right)-\tan^{-1}\left(\frac{r-\frac{1}{2}}{2}\right)\right)$$

$$= \lim_{n\to\infty}\left(\tan^{-1}\frac{3}{4}-\tan^{-1}\frac{1}{4}\right)+\left(\tan^{-1}\frac{5}{4}-\tan^{-1}\frac{3}{4}\right)+\left(\tan^{-1}\frac{7}{4}-\tan^{-1}\frac{5}{4}\right)$$
$$.........\left(\tan^{-1}\left(\frac{n}{2}+\frac{1}{4}\right)-\tan^{-1}\left(\frac{n}{2}-\frac{1}{4}\right)\right)$$

$$= \lim_{n\to\infty}\left(\tan^{-1}\left(\frac{n}{2}+\frac{1}{4}\right)-\tan^{-1}\frac{1}{4}\right) = \frac{\pi}{2}-\tan^{-1}\frac{1}{4} = \cot^{-1}\frac{1}{4} = \tan^{-1}4.$$ **Ans.**]

Time: 30 Min. **M.M.: 24** **DPP. NO.-50**

[SINGLE CORRECT CHOICE TYPE] [5 × 3 = 15]

Q.1 The slope of tangent drawn to the curve $f(x) = \sin x - \int_0^x (x-t) f(t)\, dt$ at $x = 0$, is equal to

(A*) 1 (B) 2 (C) 3 (D) 4

[Sol.$_{401/aod(tn)/SC}$ $f(0) = 0$

and $f(x) = \sin x - x\int_0^x f(t)dt + \int_0^x t\, f(t)dt$

$\therefore\ f'(0) = 1.$]

Q.2 Let $f : R \to R$ be any function. Define $g : R \to R$ by $g(x) = |f(x)|$ for all x. Then g is

(A) onto if f is onto
(B) one one if f is one one
(C*) continuous if f is continuous
(D) differentiable if f is differentiable.

[JEE 2000, Screening, 1 out of 35]

[Sol.$_{284/lcd/SC}$ $g : R \to R$ $g(x) = |f(x)|$

(1) If $f(x) = x$ which is differentiable but $g(x) = |x|$ is not diff. $\Rightarrow$ D is not
(2) ||ly $f(x) = x$ which is one-one but $|x|$ is one-one $\Rightarrow$ B is not
(3) |||ly $f(x) = x$ which is onto but g(x) is not $\Rightarrow$ A is not
however (C) is correct.]

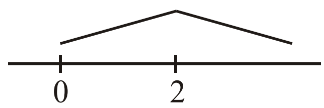
Q.3 Let $f(x) = mx - 1 + \frac{1}{x}$. Then the smallest value of the constant m such that $f(x) \geq 0$ for every $x > 0$ is

(A) $\frac{1}{8}$ (B*) $\frac{1}{4}$ (C) $\frac{1}{2}$ (D) 1

[Sol.$_{380/aod(mo)/SC}$ $m \geq \frac{1}{x} - \frac{1}{x^2}$

$\therefore m \geq f(x)]_{max}\ \forall\ x > 0$

$f'(x) = \frac{2-x}{x^3}$;



maximum occurs at $x = 2$

$m \geq f(2)$

$\therefore\ m \geq \frac{1}{4}$

$\Rightarrow$ smallest m is $\frac{1}{4}$. **Ans.**]

Q.4 If the two distinct linear functions which map the interval [–1 ,1] onto [0 , 2] are f (x) = ax + b and g (x) = cx + d then (a + b + c + d) is equal to

(A) 1 (B*) 2 (C) 3 (D) 0

[Sol.$_{142/func/SC}$ f(x) = 1 + x or 1 – x]

Q.5 Let $f : (-3, 3) \to R$ be a differentiable function with $f(0) = -2$ and $f'(0) = -1$.

and $g(x) = \left(f(3f(x)+6)\right)^3$. Then $g'(0)$ is equal to

(A) 0 (B) 9 (C*) 36 (D) – 36

[Sol.$_{145/mod/SC}$ $\because$ $g(x) = \left(f(3f(x)+6)\right)^3 \Rightarrow g'(x) = 3\left(f(3f(x)+6)\right)^2 \, f'\left(3f(x)+6\right) \cdot 3f'(x)$

$\therefore$ $g'(0) = 3\left(f(3f(0)+6)\right)^2 \cdot f'(3f(0)+6) \cdot 3f'(0)$

$= 9\left(f(-6+6)\right)^2 \, f'(-6+6) \, f'(0)$

$= 9\left(f(0)\right)^2 \left(f'(0)\right)^2 = 9 \times 4 \times 1 = 36.$]

[COMPREHENSION TYPE] [3 × 3 = 9]

Paragraph for Question 6 to 8

Let P(x) be a polynomial of degree 4, vanishes at x = 0. Given P(–1) = 55 and P(x) has relative maximum / relative minimum at x = 1, 2, 3.

Q.6 Area of triangle formed by extremum points of P(x), is

(A) $\frac{1}{2}$ (B) $\frac{1}{4}$ (C) $\frac{1}{8}$ (D*) 1

Q.7 The value of definite integral $\int_{-1}^{1} \left(P(-x) + P(x)\right) dx$, is

(A) $\frac{252}{15}$ (B*) $\frac{452}{15}$ (C) $\frac{652}{15}$ (D) $\frac{752}{15}$

Q.8 Which one of the following statement is correct?

(A) P (x) has two relative maximum points and one relative minimum point.

(B*) Range of P(x) contains 9 negative integers.

(C) Sum of real roots of P(x) = 0 is 5.

(D) P (x) has exactly one inflection point.

[Sol.$_{30602\text{-}603\text{-}604/aod}$ Let $P'(x) = A(x-1)(x-2)(x-3)$

or $P'(x) = A(x^3 - 6x^2 + 11x - 6)$

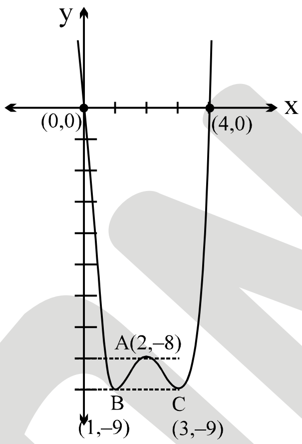
Integrating both sides with respect to x, we get

$P(x) = \frac{A}{4}(x^4 - 8x^3 + 22x^2 - 24x) + B$ .....(1)

Given, $P(0) = 0 \Rightarrow B = 0$ ......(2)

and $P(-1) = 55 \Rightarrow A = 4$ .....(3)

So, from (1), we get



$P(x) = x^4 - 8x^3 + 22x^2 - 24x$

$= x(x^3 - 8x^2 + 22x - 24)$

$= x(x-4)\underbrace{(x^2 - 4x + 6)}_{\text{non-real roots}}$

(i) Clearly, area of triangle formed by extremum points of $P(x) = \frac{1}{2} \times 2 \times 1 = 1$

(ii) As, $P(x) + P(-x) = \underbrace{2x^4 + 44x^2}_{\text{even function}}$

So, $\int_{-1}^{1} (P(-x) + P(x))dx = 2\int_0^1 (2x^4 + 44x^2)dx = 4\int_0^1 (x^4 + 22x^2)dx = 4\left[\frac{x^5}{5} + \frac{22x^3}{3}\right]_0^1$

$= 4\left[\frac{1}{5} + \frac{22}{3}\right] = 4\left[\frac{3+110}{15}\right] = \frac{4 \times 113}{15} = \frac{452}{15}$ **Ans.**

(iii) From above graph, P(x) has two relative minimum points and one relative maximum point
$\Rightarrow$ (A) is incorrect
Clearly, from above graph, range of $P(x) = [-9, \infty)$ $\Rightarrow$ (B) is correct
As, real roots of equation $P(x) = 0$ are 0 and 4. So sum of real roots = 4 $\Rightarrow$ (C) is incorrect
As, $P'(x) = 0$ has 3 distinct real roots, so $P''(x) = 0$ has two distinct real roots.
$\Rightarrow$ P(x) has two inflection points $\Rightarrow$ (D) is incorrect.]

Time: 30 Min. **M.M.: 20** **DPP. NO.-51**

[MULTIPLE CORRECT CHOICE TYPE] [5 × 4 = 20]

Q.1 Let $I = \int_0^1 \sqrt{\dfrac{1+\sqrt{x}}{1-\sqrt{x}}}\,dx$ and $J = \int_0^1 \sqrt{\dfrac{1-\sqrt{x}}{1+\sqrt{x}}}\,dx$, then the correct statement is

(A) $I + J = 2$ (B*) $I - J = \pi$ (C) $I = \dfrac{2+\pi}{2}$ (D*) $J = \dfrac{4-\pi}{2}$

[Sol.$_{\text{40857/def/MORE}}$ $I + J = \int_0^1 \dfrac{(1+\sqrt{x})+(1-\sqrt{x})}{\sqrt{1-x}}\,dx = \int_0^1 \dfrac{2}{\sqrt{1-x}}\,dx$ Put $x = \sin^2\theta$

$$I + J = \int_0^{\pi/2} \frac{2\cdot 2\sin\theta\cos\theta}{\cos\theta}\,d\theta = 4\int_0^{\pi/2}\sin\theta\,d\theta = 4$$

$$I - J = \int_0^1 \frac{2\sqrt{x}}{\sqrt{1-x}}\,dx$$ Put $x = \sin^2\theta$

$$I - J = \int_0^{\pi/2} \frac{2\sin\theta\,2\sin\theta\cos\theta}{\cos\theta}\,d\theta = 4\int_0^{\pi/2}\sin\theta\,d\theta = \pi.$$ **Ans.**]

Q.2 If tangents are drawn from a point P (2, 0) to the curve $\sqrt{1+y^2} = \dfrac{x}{3}$ which meet the curve at A and B, then

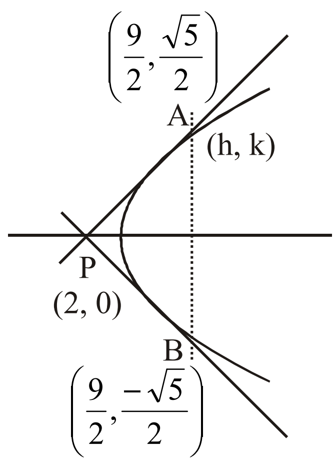
(A*) Acute angle between the tangents is $\tan^{-1}\left(\dfrac{\sqrt{5}}{2}\right)$

(B*) $AB = \sqrt{5}$

(C*) Area of $\Delta PAB = \dfrac{5\sqrt{5}}{4}$

(D) ΔPAB is equilateral triangle.

[Sol.$_{\text{40002/aod(tn)/MORE}}$ $\dfrac{2y}{2\sqrt{1+y^2}}\left(\dfrac{dy}{dx}\right) = \dfrac{1}{3}$



$$\frac{dy}{dx} = \frac{\sqrt{1+y^2}}{3y}$$

$$\left.\frac{dy}{dx}\right|_{(h,k)} = \frac{\sqrt{1+k^2}}{3k}$$

$$\therefore\ \frac{k-0}{h-2} = \frac{\sqrt{1+k^2}}{3k} \Rightarrow \sqrt{1+k^2} = \frac{h}{3} \Rightarrow 1 + k^2 = \frac{h^2}{9}$$

$$\frac{3k^2}{h-2} = \sqrt{1+k^2} \Rightarrow \frac{3\left(\frac{h^2}{9}-1\right)}{h-2} = \frac{h}{3} \Rightarrow \frac{h^2-9}{h-2} = h$$

$$\Rightarrow h^2 - 9 = h^2 - 2h \Rightarrow h = \frac{9}{2}; k = \pm\frac{\sqrt{5}}{2}.$$

(A) Angle between tangents $\theta = 2\tan^{-1}\left(\frac{1}{\sqrt{5}}\right) = \tan^{-1}\left(\frac{2\cdot\frac{1}{\sqrt{5}}}{1-\frac{1}{5}}\right) = \tan^{-1}\left(\frac{\sqrt{5}}{2}\right)$.

(B) $AB = \sqrt{5}$

(C) Area of $\Delta PAB = \frac{1}{2}\times\frac{5}{2}\times\sqrt{5} = \frac{5\sqrt{5}}{4}$ sq. units.

(D) ΔPAB is not equilateral.]

Q.3 The function, $f(x) = \max\{(1-x), (1+x), 2\}, x \in (-\infty, \infty)$ is :
(A*) Continuous at all points
(B) Differentiable at all points
(C*) Differentiable at all points except at $x = 1$ & $x = -1$
(D) Continuous at all points except at $x = 1$ & $x = -1$, where it is discontinuous. [JEE'95, 2]

[Sol.$_{\text{40615/lcd/MORE}}$]

Q.4 Let f be a continuous function on R and satisfies $f(x) = e^x + \int_0^1 e^x f(t)\,dt$,
then which of the following is(are) correct?
(A*) $f(0) < 0$
(B*) $f(x)$ is decreasing function on R
(C) $f(x)$ is an increasing function on R
(D) $\int_0^1 f(x)\,dx > 0$

[Sol.$_{\text{40894/aod/MORE}}$ We have $f(x) = e^x + \int_0^1 e^x f(t)\,dt = e^x + ke^x$, where $k = \int_0^1 f(t)\,dt$

$\therefore \quad k = \int_0^1 \left(e^t + ke^t\right)dt = e + ke - 1 - k$

So, $\quad k = \frac{e-1}{2-e}$

$\therefore \quad f(x) = e^x\left(1+\frac{e-1}{2-e}\right) = \frac{e^x}{2-e}$

Obviously, $f(0) = \frac{1}{2-e} < 0$

Also $f'(x) = \frac{e^x}{2-e} < 0 \ \forall x \in R$

Hence, $f(x)$ is a decreasing function on R.

Also $\int_0^1 f(x)\,dx = \int_0^1 \frac{e^x}{2-e}dx = \left[\frac{e^x}{2-e}\right]_0^1 = \frac{e-1}{2-e} < 0$]

Q.5 Let $h(x) = \min\{x, x^2\}$, for every real number of x. Then :

(A*) h is cont. for all x
(B) h is diff. for all x
(C*) $h'(x) = 1$, for all $x > 1$
(D*) h is not diff. at two values of x. [JEE'98, 2]

[Sol.$_{40616/lcd/MORE}$]

Time: 30 Min. **M.M.: 25** **DPP. NO.-52**

[INTEGER TYPE] [5 × 5 = 25]

Q.1 Let d be the number of integers in the range of the function $f(x) = \begin{cases} 4, & \text{if } -4 \le x < -2 \\ |x|, & \text{if } -2 \le x < 7 \\ \sqrt{x}, & \text{if } 7 \le x < 14 \end{cases}$.

Also roots of $P(x) = x^2 + mx - 4m + 20$ are α and β. If $\alpha < \frac{d-3}{4} < \frac{d-3}{2} < \beta$ and the smallest integral value of m is k, then find the value of $(k - 5)$. [Ans. 8]

[Sol.$_{50003/func/OMR}$ Range of f(x) is [0, 7)

Hence, d = 7

Now, one root of P(x) is less than 1 and other root greater than 2.

Hence, $P(1) < 0 \Rightarrow 21 - 3m < 0 \Rightarrow m > 7$

and $P(2) < 0 \Rightarrow 24 - 2m < 0 \Rightarrow m > 12$

Hence, m > 12.

$\therefore$ Least integral value of m is 13

$\Rightarrow (k - 5) = 8$. **Ans.**]

Q.2 If y is a function of x and $\ln(x + y) = 2xy$, then find the value of $y''(0)$. [Ans. 8]

[Sol.$_{50717/mod}$ $\ln(x + y) = 2xy$

x = 0, y = 1

$$\frac{1+y'}{x+y} = 2(xy' + y)$$

Put x = 0, y = 1 $\qquad 1 + y' = 2(0 + 1) = 2$

$\Rightarrow y' = 1$

$$\frac{(x+y)y''-(1+y')^2}{(x+y)^2} = 2(xy'' + 2y')$$

x = 0, y = 1, y' = 1

$$\frac{y''-4}{1} = 2(0 + 2) = 4$$

$\Rightarrow y''(0) = 8$]

Q.3 Let $f(x)=\dfrac{1}{e^x+8e^{-x}+4e^{-3x}}$ and $g(x)=\dfrac{1}{e^{3x}+8e^x+4e^{-x}}$. If $\int(f(x)-2g(x))dx=h(x)+c$, where c is constant of integration and $\lim\limits_{x\to\infty}h(x)=\dfrac{\pi}{4}$, then find the value of $2\tan(2h(0))$.

[Ans. 3]

[Sol.$_{50726/\text{inde/OMR}}$ $f(x)=\dfrac{e^{3x}}{e^{4x}+8e^{2x}+4}$ and $g(x)=\dfrac{e^x}{e^{4x}+8e^{2x}+4}$

$$\text{Integral}=\int(f(x)-2g(x))dx=\int\frac{(e^{3x}-2e^x)}{e^{4x}+8e^{2x}+4}dx$$

Let $e^x=t$

$$I=\int\frac{(t^2-2)}{t^4+8t^2+4}\,dt=\int\frac{\left(1-\frac{2}{t^2}\right)dt}{t^2+8+\frac{4}{t^2}}$$

$$=\int\frac{\left(1-\frac{2}{t^2}\right)dt}{\left(t+\frac{2}{t}\right)^2+4}=\frac{1}{2}\tan^{-1}\left(\frac{t+\frac{2}{t}}{2}\right)+c=\frac{1}{2}\tan^{-1}\left(\frac{e^x+2e^{-x}}{2}\right)+c$$

$$\therefore\quad h(x)=\frac{1}{2}\tan^{-1}\left(\frac{e^x+2e^{-x}}{2}\right)$$

$$\therefore\quad h(0)=\frac{1}{2}\tan^{-1}\left(\frac{3}{2}\right)\Rightarrow 2\tan(2h(0))=3$$ **Ans.**]

Q.4 If $\lim\limits_{n\to\infty}\dfrac{e\left(1-\frac{1}{n}\right)^n-1}{n^\alpha}$ exists and is equal to non-zero constant c, then find the value of $12(c-\alpha)$.

[Ans. 6]

[Sol.$_{50781/\text{lcd/OMR}}$ $\lim\limits_{n\to\infty}\dfrac{e\,e^{n\,ln\left(1-\frac{1}{n}\right)}-1}{n^\alpha}$

$$\lim_{n\to\infty}\frac{\left(e^{\overbrace{n\,ln\left(1-\frac{1}{n}\right)+1}^{M}}-1\right)}{n^\alpha}=\lim_{n\to\infty}\frac{(e^M-1)}{M}\cdot\frac{M}{n^\alpha}$$

$$\lim_{n\to\infty}\frac{n\ ln\left(1-\frac{1}{n}\right)+1}{n^{\alpha}}=c$$

$$\lim_{n\to\infty}\frac{n\left[\frac{-1}{n}-\frac{1}{n^2}\frac{1}{2}+......\right]+1}{n^{\alpha}}$$

for limit to exist $\alpha = -1$.

$\therefore$ Limit is $\frac{-1}{2}=c \Rightarrow c-\alpha=\frac{-1}{2}-(1)=\frac{1}{2}$

$\therefore$ $12(c-\alpha)=6$. **Ans.**]

Q.5 Let $f(x)=12\left(\frac{e^{3x}-3e^{x}}{e^{2x}-1}\right)$ be defined for $x>0$ and $g(x)$ be the inverse of $f(x)$.

If $\int_{8}^{27} g(x)\,dx = a\,ln\,3 - b\,ln\,2 - c$, then find the value of $a-(b+c)$. [Ans. 7]

[Sol.$_{50139/def/OMR}$ $\int_{ln 2}^{ln 3} f(x)\,dx + \int_{8}^{27} g(y)\,dy = 27\,ln\,3 - 8\,ln\,2$

and $\int_{ln 2}^{ln 3} f(x)\,dx = 12 - 12\,ln\,3 + 12\,ln\,2$

$\therefore \int_{8}^{27} g(y)\,dy = 39 ln\,3 - 20\,ln\,2 - 12$

Hence, a = 39; b = 20; c = 12

$\therefore$ a – (b + c) = 39 – 32 = 7. **Ans.**]