Integrals

Integration

It is the inverse of differentiation. Let, $\frac{d}{dx}F(x)=f(x)$. Then, $\int f(x)dx=F(x)+c$, 'c' : constant of integral. These integrals are called indefinite or general integrals.

Properties of indefinite integrals are

(i) $\int[f(x)+g(x)]dx=\int f(x)dx+\int g(x)dx$, (ii) $\int kf(x)dx=k\int f(x)dx$,

For eg : $\int(3x^2+2x)dx=x^3+x^2+c$ where k is real.

Some Standard integrals

(i) $\int x^n dx=\frac{x^{n+1}}{n+1}+c, n\neq -1$ like, $\int dx=x+c$

(ii) $\int \cos x\,dx=\sin x+c$ (iii) $\int \sin x\,dx=-\cos x+c$

(iv) $\int \sec^2 x\,dx=\tan x+c$ (v) $\int \operatorname{cosec}^2 x\,dx=-\cot x+c$

(vi) $\int \sec x\tan x\,dx=\sec x+c$ (vii) $\int \operatorname{cosec}x\cot x\,dx=-\operatorname{cosec}x+c$

(viii) $\int\frac{dx}{\sqrt{1-x^2}}=\sin^{-1}x+c$ (ix) $\int\frac{dx}{\sqrt{1-x^2}}=-\cos^{-1}x+c$

(x) $\int\frac{dx}{1+x^2}=\tan^{-1}x+c$ (xi) $\int\frac{dx}{1+x^2}=-\cot^{-1}x+c$

(xii) $\int e^x dx=e^x+c$ (xiii) $\int a^x dx=\frac{a^x}{\log a}+c$

(xiv) $\int\frac{dx}{x\sqrt{x^2-1}}=\sec^{-1}x+c$ (xv) $\int\frac{dx}{x\sqrt{x^2-1}}=-\operatorname{cosec}^{-1}x+c$

(xvi) $\int\frac{1}{x}dx=\log|x|+c$

Integration by substitution

The method in which we change the variable to some other variable is called the method of substitution

$\int\tan x\,dx=\log|\sec x|+c$ $\int\cot x\,dx=\log|\sin x|+c$

$\int\sec x\,dx=\log|\sec x+\tan x|+c$ $\int\operatorname{cosec}x\,dx=\log|\operatorname{cosec}x-\cot x|+c.$

Integration of some special functions

(i) $\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c$ (ii) $\int\frac{dx}{a^2-x^2}=\frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+c$

(iii) $\int\frac{dx}{x^2+a^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}+c$ (iv) $\int\frac{dx}{\sqrt{x^2-a^2}}=\log\left|x+\sqrt{x^2-a^2}\right|+c$

(v) $\int\frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\frac{x}{a}+c$ (vi) $\int\frac{dx}{\sqrt{x^2+a^2}}=\log\left|x+\sqrt{x^2+a^2}\right|+c.$

Some special type of integrals

(i) $\int\sqrt{x^2-a^2}\,dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+c.$

(ii) $\int\sqrt{x^2+a^2}\,dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+c$

(iii) $\int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+c.$

First fundamental theorem of integral calculus

Let the area function be defined by $A(x)=\int_a^x f(x)dx\ \forall x\geq a$, where f is continuous on $[a,b]$ then $A'(x)=f(x)\ \forall x\in[a,b]$.

Integration by parts

$$\int f_1(x)f_2(x)dx=f_1(x)\int f_2(x)dx-\int\left[\frac{d}{dx}f_1(x)\int f_2(x)dx\right]dx$$

Second fundamental theorem of integral Calculus

Let f be a continuous function of x defined on $[a,b]$ and let F be another function such that $\frac{d}{dx}F(x)=f(x)\ \forall x\in$ domain of f, then $\int_a^b f(x)dx=[F(x)+c]_a^b=F(b)-F(a)$. This is called the definite integral of f over the range $[a,b]$, where a and b are called the limits of integration, a being the lower limit and b be the upper limit.

Integration by partial fractions

A rational function of the form $\frac{P(x)}{Q(x)}(Q(x)\neq 0)=T(x)+\frac{P_1(x)}{Q(x)}$, $P_1(x)$ has degree less than that of $Q(x)$. We can integrate $\frac{P_1(x)}{Q(x)}$ by expressing it in the following forms –

(i) $\frac{px+q}{(x-a)(x-b)}=\frac{A}{x-a}+\frac{B}{x-b}, a\neq b.$

(ii) $\frac{px+q}{(x+a)^2}=\frac{A}{x-a}+\frac{B}{(x-a)^2}$

(iii) $\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}=\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$

(iv) $\frac{px^2+qx+r}{(x-a)^2(x-b)}=\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{x-b}$

(v) $\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}=\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$

Example

$$\int_{-\pi/4}^{\pi/4}\sin^2 x\,dx$$

$$=2\int_0^{\pi/4}\sin^2 x\,dx$$

$$=2\int_0^{\pi/4}\left(\frac{1-\cos 2x}{2}\right)dx$$

$$=\int_0^{\pi/4}(1-\cos 2x)dx$$

$$=\left[x-\frac{\sin 2x}{2}\right]_0^{\pi/4}$$

$$=\frac{\pi}{4}-\frac{1}{2}$$