Applications of Derivatives

Approximations

Let $y=f(x)\Delta x$ be a small increment in 'x' and Δy be the small increment in y corresponding to the increment in 'x', i.e. $\Delta y=f(x+\Delta x)-f(x)$. Then, Δy is given by $dy=f'(x)dx$ or $dy=\left(\frac{dy}{dx}\right)\Delta x$, is a good approximation of Δy when $dx=\Delta x$ is relatively small and denote by $dy\approx\Delta y$. For eg: Let us approximate $\sqrt{36.6}$. To do this, we take $y=\sqrt{x}, x=36$, $\Delta x=0.6$ then $\Delta y=\sqrt{x+dx}-\sqrt{x}$

$=\sqrt{36.6}-\sqrt{36}$

$=\sqrt{36.6}-6 \Rightarrow \sqrt{36.6}=6+dy$

Now, dy is approximately Δy and is given by dy

$=\left(\frac{dy}{dx}\right)\Delta x=\frac{1}{2\sqrt{x}}(0.6)=\frac{1}{2\sqrt{36}}(0.6)=0.05$. So, $\sqrt{36.6}\approx 6+0.05=6.05$.

Rate of Change Quantities

If a quantity if 'y' varies with another quantity x so that $y = f(x)$, then $\frac{dy}{dx}[f'(x)]$ represents the rate of change of y w.r.t x and $\left.\frac{dy}{dx}\right|_{x=x_0}(f'(x_0))$ represents the rate of change of y w.r.t. x at $x=x_0$

If 'x' and 'y' varies with another variable 't' i.e., if $x=f(t)$ and $y=g(t)$, then by chain rule $\frac{dy}{dx}=\frac{dy}{dt}\Big/\frac{dx}{dt}$, if $\frac{dx}{dt}\neq 0$.

For eg: if the radius of a circle, r = 5 cm, then the rate of change of the area of a circle per second w.r.t 'r' is – $\left.\frac{da}{dr}\right|_{r=5}=\frac{d}{dr}(\pi r^2)\Big|_{r=5}=2\pi r\Big|_{r=5}=10\pi$

Maximum and Minima

A point C in the domain of 'f' at which either $f'(c)=0$ or is not differentiable is called a critical point of f.

Second Derivative test

Let f be a function defined on I and C∈I, f is twice differentiable at C. Then **(i)** $x=C$ is a point of local max. If $f'(C)=0$ and $f''(C)<0$, $f(C)$ is local max. of f. **(ii)** $x=C$ is a point of local min if $f'(C)=0$ and $f''(C)>0$. $f(C)$ is local min of f. **(iii)** The test fails if $f'(C)=0$ and $f''(C)=0$

First Derivative test

Let f be continuous at a critical point C in open I. Then **(i)** if $f'(x)>0$ at every point left of C and $f'(x)<0$ at every point right of C, then 'C' is a point of local maxima. **(ii)** If $f'(x)<0$ at every point left of C and $f'(x)>0$ at every point right of C, then 'C' is a point of local minima. **(iii)** If $f'(x)$ does not change sign as 'x' increases through C, then 'C' is called the point of inflection.

Increasing and decreasing functions

A function f is said to be (i) increasing on (a,b) if $x_1<x_2$ in $(a,b)\Rightarrow f(x_1)\le f(x_2)\forall x_1,x_2\in(a,b)$, and (ii) decreasing on (a,b) if $x_1<x_2$ in $(a,b)\Rightarrow f(x_1)>f(x_2)\forall x_1,x_2\in(a,b)$

If $f'(x)\ge 0\,\forall x\in(a,b)$ then f is increasing in (a,b) and if $f'(x)\le 0\,\forall x\in(a,b)$, then f is decreasing in (a,b) For eg: Let $f(x)=x^3-3x^2+4x, x\in R$, then $f'(x)=3x^2-6x+4=3(x-1)^2+1>0\,\forall x\in R$. So, the function f is strictly increasing on R.

Tangents and Normals

The equation of the tangent at (x_0,y_0), to the curve $y=f(x)$ is given by $(y-y_0)=\left.\frac{dy}{dx}\right]_{(x_0,y_0)}(x-x_0)$ if $\frac{dy}{dx}$ does not exists at (x_0,y_0), then the tangent at (x_0,y_0) is parallel to the y–axis and its equation is $x=x_0$. If tangent to a curve $y=f(x)$ at $x=x_0$ is parallel to x–axis, then $\left.\frac{dy}{dx}\right|_{x=x_0}=0$.

Equation of the normal to the curve

$y=f(x)$ at (x_0,y_0) is $y-y_0=-\frac{1}{\left.\frac{dy}{dx}\right|_{(x_0,y_0)}}(x-x_0)$ if $\frac{dy}{dx}$ at (x_0,y_0) is zero, then equation of the normal is $x=x_0$. If $\frac{dy}{dx}$ at (x_0,y_0) does not exist, then the normal is parallel to x-axis and its equation is $y=y_0$ For eg: Let $y=x^3-x$ be a curve, then the slope of the tangent to $y=x^3-x$ at $x=2$ is $\left.\frac{dy}{dx}\right|_{x=2}=3x^2-1=3.2^2-1=11$