Determinants

Minors and cofactors of a matrix

Minor of an element a_{ij} in a determinant of matrix A is the determinant obtained by deleting i^{th} row and j^{th} column and is denoted by M_{ij}. If M_{ij} is the minor of a_{ij} and cofactor of a_{ij} is A_{ij} given by $A_{ij} = (-1)^{i+j} M_{ij}$.

- If $A_{3\times3}$ is a matrix, then $|A| = a_{11}.A_{11} + a_{12}.A_{12} + a_{13}.A_{13}$.
- If elements of one row (or column) are multiplied with cofactors of elements of any other row (or column), then their sum is zero. For *e.g.*, $a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{33} = 0$.

e.g., if $A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$, then $M_{11} = 4$ and $A_{11} = (-1)^{1+1}4 = 4$.

Determinant of square matrix 'A' |A| is given by

(i) if $A = [a_{11}]_{1\times1}$, then $|A| = a_{11}$

(ii) if $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times2}$ then $|A| = a_{11}a_{22} - a_{12}a_{21}$

(iii) if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times3}$, then $|A| = a_{11}(a_{22}.a_{33} - a_{23}.a_{32}) - a_{12}(a_{21}.a_{33} - a_{23}.a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$

For *eg.* if $A = \begin{bmatrix} 2 & 3 \\ 2 & 4 \end{bmatrix}$, then $|A| = 2\times4 - 3\times2 = 2$

Properties of |A|

(i) $|A|$ remains unchanged, if the rows and columns of A are interchanged ie., $|A| = |A'|$

(ii) if any two rows (or columns) of A are interchanged, then the sign of $|A|$ changes.

(iii) if any two rows (or columns) of A are identical, then $|A| = 0$

(iv) if each element of a row (or a column) of A is multiplied by B (const.), then $|A|$ gets multiplied by B.

(v) if $A = [a_{ij}]_{3\times3}$, then $|k.A| = k^3|A|$.

(vi) if elements of a row or a column in a determinant $|A|$ can be expressed as sum of two or more elements, then $|A|$ can be expressed as $|B| + |C|$.

(vii) if $R_i \to R_i + kR_j$ or $C_i = C_i + kC_j$ in $|A|$, then the value of $|A|$ remains same

Adjoint and inverse of a matrix

if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then adj. $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$, where A_{ij} is the cofactor of a_{ij}.

- $A(\text{adj}.A) = (\text{adj}.A).A = |A| = I$, A – square matrix of order 'n'
- if $|A| = 0$, then A is singular. Otherwise, A is non-singular.
- if $AB = BA = I$, where B is a square matrix, then B is called the inverse of A, $A^{-1} = B$ or $B^{-1} = A$, $(A^{-1})^{-1} = A$.

Inverse of a square matrix exists if A is non-singular *i.e.* $|A| \neq 0$, and is given by

$$A^{-1} = \frac{1}{|A|}(\text{adj}.A)$$

Area of triangle

if (x_1, y_1), (x_2, y_2) and (x_3, y_3) $\Delta = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

For eg: if (1, 2), (3, 4) and (–2, 5) are the vertices, then area of the triangle is

$$\Delta = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 4 & 1 \\ -2 & 5 & 1 \end{vmatrix} = 1(4-5) - 2(3+2) + 1(15+8) = 12 \text{ sq.units.}$$

we take positive value of the determinant.

Applications of Determinants

- if $a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$, $a_3x + b_3y + c_3z = d_3$ then we can write $AX = B$, where $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$
- Unique solution of $AX = B$ is $X = A^{-1}B$, $|A| \neq 0$.
- $AX = B$ is consistent or inconsistent according as the solution exists or not.
- For a square matrix A in $AX = B$, if
  (i) $|A| \neq 0$ then there exists unique solution.
  (ii) $|A| = 0$ and (adj. A) $B \neq 0$, then no solution.
  (iii) if $|A| = 0$ and (adj.A).$B = 0$ then system may or may not be consistent.